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Decay in a uniform field: an exactly solvable model

R M Cavalcanti[1], P Giacconi[2] and R Soldati[2]

[1] Instituto de Física, Universidade Federal do Rio de Janeiro, Caixa Postal 68528, 21941-972 Rio de Janeiro, RJ, Brazil
[2] Dipartimento di Fisica, Università di Bologna, Istituto Nazionale di Fisica Nucleare, Sezione di Bologna, 40126 Bologna, Italy

E-mail: rmoritz@if.ufrj.br, giacconi@bo.infn.it and soldati@bo.infn.it

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Abstract
We investigate the time evolution of the decay (or ionization) probability of a D-dimensional model atom ($D = 1, 2, 3$) in the presence of a uniform (i.e., static and homogeneous) background field. The model atom consists in a non-relativistic point particle in the presence of a point-like attractive well. It is shown that the model exhibits infinitely many resonances leading to possible deviations from the naive exponential decay law of the non-decay (or survival) probability of the initial atomic quantum state. Almost stable states exist due to the presence of the attractive interaction, no matter how weak it is. Analytic estimates as well as numerical evaluation of the decay rates are explicitly given and discussed.

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1. Introduction

Exponential decay is a common feature of many physical processes; in particular, it is the universal hallmark of unstable systems such as radioactive nuclei. However, it is known that under very general conditions quantum mechanics predicts deviations from the exponential decay within short as well as long time intervals [1]. As pointed out by Khalfin [2], the latter situation occurs whenever the spectrum of the Hamiltonian H is bounded from below; in this case, the Paley–Wiener theorem [3] on Fourier transforms implies that the non-decay or survival amplitude $A(t; [\psi]) := \langle\psi(0)|\psi(t)\rangle$ necessarily satisfies

$$\int_{-\infty}^{+\infty} \frac{|\ln|A(t;[\psi])||}{1+t^2}\,\mathrm{d}t < \infty. \tag{1}$$

This condition clearly rules out an exponential decay for $t \to \infty$, as this would cause the integral above to diverge[3]. Explicit calculations in a number of models [4] show that, in fact, there occurs a crossover from exponential to power law decay when $t \to \infty$.

One may wonder what happens if the Hamiltonian is not bounded from below. At first glance this might appear to be an academic question, since any realistic Hamiltonian should be bounded from below. However, such 'unrealistic' Hamiltonians are often found in physics. Some examples are the decay of a metastable vacuum through the formation of bubbles of the true vacuum [5], the droplet model for first-order phase transitions in statistical physics [6], or the ionization of an atom by a static electric field [7]. In the latter case Herbst [8] provided a partial answer to that question. Let $H = -\Delta + V + Fx$ be the Hamiltonian describing a one-electron atom in a uniform electric field. If $V(x, y, z)$ is holomorphic in x and $V(x + \mathrm{i}a, y, z)$ is bounded and decreases to zero as $r := (x^2 + y^2 + z^2)^{1/2} \to \infty$ for each $a \in \mathbb{R}$, whereas ψ is an eigenvector of $-\Delta + V$ with negative eigenvalue, then (for $F \neq 0$)

$$\langle\psi|\exp(-\mathrm{i}Ht)|\psi\rangle = \sum_{\Gamma_j \leqslant \alpha} C_j \exp(-\mathrm{i}E_j t) + O(\exp(-(\alpha + \varepsilon)t/2)) \tag{2}$$

where E_j are the resonances of H (i.e., the complex poles of $(E - H)^{-1}$ in the lower half-plane) whereas $\Gamma_j := -2\,\mathrm{Im}\,E_j$ are their widths and α and ε are suitable positive numbers. Herbst also showed that $\inf\{\Gamma_j\} > 0$, in such a way that equation (2) ensures the exponential decay of $A(t; \psi)$ as $t \to \infty$.

The purpose of the present paper is to investigate the decay law in a very simple (albeit non-trivial) model, namely a one-electron atom in which the Coulomb attractive potential is replaced by a point-like attractive well—an idealization of a very short-range attractive interaction—and put under the influence of a static and uniform electric or gravitational field (for related simple models, see [9, 10]). The model will be studied in $D = 1, 2, 3$ space dimensions and will be shown to be exactly solvable in the one- and three-dimensional cases, whereas in the two-dimensional case it is solvable up to a quadrature. In spite of its simplicity, this model unravels some remarkable features that can actually be evaluated in detail and become worthwhile to be used as a paradigm with respect to more realistic situations, without any substantial change in the basic physical contents. With this concern, it is known that, in the absence of a uniform field, this model exhibits a bound state (see for instance [11–13]).

It turns out that, once a background uniform field has been switched on, an infinite number of resonances arise in this model. In particular, the state vector that corresponds to the bound state in the absence of a uniform field is turned into a *bona fide* quasi-stable state for a sufficiently weak external field. For instance, if the bound state energy is of the order of 1 eV, the lifetime of the corresponding quasi-stable state in the presence of the Earth's gravitational field is much longer than the present age of the Universe; even in the presence of a rather strong laboratory static electric field, its lifetime is long in comparison to the typical time scales of atomic and condensed matter physics. On the other hand, very strong external fields are expected to create non-perturbative deviations from the naive exponential decay law. This has been observed in previous numerical studies of the present model [14–16] and will be qualitatively explained in this work.

The paper is organized as follows. In section 2 we first analyse the one-dimensional case, where a direct one-to-one correspondence takes place between the strength of the attractive potential well and the bound state energy. All the main features of the model are explicitly

[3] Another measure of decay, used for states initially confined inside a region $\mathcal{M}$ (i.e., $\psi(x, t = 0)$ vanishes outside $\mathcal{M}$), is the non-escape probability, defined as $P(t) := \int_{\mathcal{M}} |\psi(x, t)|^2\,\mathrm{d}x$. Using Schwarz's inequality one can easily show that $P(t) \geqslant |A(t)|^2$, so that Khalfin's argument also rules out the exponential decay of $P(t)$ for $t \to \infty$.

exhibited and discussed. In section 3 we generalize our investigation to the two- and three-dimensional cases. Here the renormalization procedure is mandatory, in order to remove the ultraviolet divergences of the Green functions. In so doing, the bound state energy in the zero-field case achieves a deeper physical meaning—it specifies the self-adjoint extension of the quantum Hamiltonian operator—whereas the renormalized coupling parameters become *running* auxiliary quantities. *Mutatis mutandis*, all the main physical properties of the one-dimensional case are essentially recovered. In section 4 we draw our conclusions, whilst we defer some technical although important details to the appendixes.

2. The one-dimensional model

Let us consider the Hamiltonian[4]

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \lambda\delta(x) - Fx \qquad \lambda > 0 \quad F > 0 \tag{3}$$

describing a particle interacting with an attractive δ-potential and a uniform background field. In the absence of the field (i.e., when $F = 0$) there is a single bound state with energy $E_B = -\lambda^2/4$, the corresponding wavefunction being given by $\psi_B(x) = (\lambda/2)^{1/2}\exp(-\lambda|x|/2)$. Once the uniform field is turned on, this bound state becomes unstable, in the sense that $A(t; [\psi_B]) \to 0$ as $t \to \infty$. The precise way in which this occurs will be the subject of this section.

2.1. Retarded Green function

The retarded Green function $G^+(E; x, x')$ is the solution to the differential equation

$$(E - H)G^+(E; x, x') = \delta(x - x') \qquad E \in \mathbb{C} \tag{4}$$

that satisfies the boundary condition

$$\lim_{|x|\to\infty} G^+(E; x, x') = 0 \qquad \text{for} \quad \mathrm{Im}(E) > 0; \tag{5}$$

it is defined for $\mathrm{Im}(E) \leqslant 0$ by analytic continuation. The solution to equation (4) is known [12, 16, 17], but we shall derive it here for the sake of completeness.

To solve equation (4), let us first consider the case $\lambda = 0$; it can then be rewritten as

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \rho\right) G_0^+(\rho, \rho') = F^{-1/3}\delta(\rho - \rho') \tag{6}$$

where

$$\rho := F^{1/3}\left(x + \frac{E}{F}\right). \tag{7}$$

The solution to equation (6) that satisfies the boundary condition (5) is given by

$$G_0^+(\rho, \rho') = a\mathrm{Ai}(-\rho)\theta(\rho' - \rho) + b\,\mathrm{Ci}^{(+)}(-\rho)\theta(\rho - \rho') \tag{8}$$

where $\mathrm{Ai}(x)$ and $\mathrm{Ci}^{(+)}(x) := \mathrm{Bi}(x) + \mathrm{i}\mathrm{Ai}(x)$ are Airy functions [18] and $\theta(x)$ is the Heaviside step function. The coefficients a and b are fixed by the matching conditions at $\rho = \rho'$:

$$G_0^+(\rho' + 0, \rho') = G_0^+(\rho' - 0, \rho') \tag{9}$$

$$\partial_\rho G_0^+(\rho, \rho')|_{\rho=\rho'+0} - \partial_\rho G_0^+(\rho, \rho')|_{\rho=\rho'-0} = F^{-1/3}. \tag{10}$$

[4] We use atomic units such that $\hbar = 2m = 1$.

Solving these equations one finally arrives at

$$G_0^+(\rho,\rho') = -\pi F^{-1/3}\mathrm{Ai}(-\rho_-)\,\mathrm{Ci}^{(+)}(-\rho_+) \tag{11}$$

where $2\rho_\pm := \rho + \rho' \pm |\rho - \rho'|$.

In order to obtain $G^+(E; x, x')$ for $\lambda \neq 0$, we rewrite equation (4) as an integral equation

$$\begin{aligned} G^+(E; x, x') &= G_0^+(E; x, x') - \int_{-\infty}^{+\infty} \mathrm{d}y\, G_0^+(E; x, y)\lambda\delta(y)G^+(E; y, x') \\ &= G_0^+(E; x, x') - \lambda G_0^+(E; x, 0)G^+(E; 0, x'). \end{aligned} \tag{12}$$

Taking $x = 0$, solving for $G^+(E; 0, x')$, and reinserting the result into equation (12) yields the so-called Krein formula [11]:

$$G^+(E; x, x') = G_0^+(E; x, x') - \frac{G_0^+(E; x, 0)G_0^+(E; 0, x')}{g(\lambda, E)} \tag{13}$$

where

$$g(\lambda, E) := \frac{1}{\lambda} + G_0^+(E; 0, 0). \tag{14}$$

2.2. *Resonant-mode expansion of the propagator*

From $G^+(E; x, x')$ one can obtain the retarded propagator $K^+(t; x, x')$ by a Fourier transformation:

$$K^+(t; x, x') = \mathrm{i}\int_{-\infty}^{+\infty} \frac{\mathrm{d}E}{2\pi}\,\mathrm{e}^{-\mathrm{i}Et}G^+(E; x, x'). \tag{15}$$

It turns out that the following bound on $G^+(E; x, x')$ holds true in the lower half-plane[5] for $|E|$ sufficiently large (see appendix A):

$$|G^+(E; x, x')| \lesssim C|E|^{-1/2}\exp\{|E|^{1/2}(|x| + |x'|)\} \qquad |E| \to \infty \tag{16}$$

where C is a suitable constant.

This bound allows one to close the contour of integration of (15) when $t > 0$ with a semi-circle of infinite radius in the lower half-plane without changing the value of the integral. Using Cauchy's theorem, one then obtains the so-called resonant-mode expansion of the propagator [19, 20]:

$$K^+(t; x, x') = \sum_n \exp(-\mathrm{i}E_n t)\varphi_n(x)\varphi_n(x') \tag{17}$$

where the sum runs over the poles[6] of $G^+(E; x, x')$ located in the lower half-plane and the functions $\varphi_n(x)$ are given by

$$\varphi_n(x) = \left.\frac{G_0^+(E; x, 0)}{\left[-\partial_E G_0^+(E; 0, 0)\right]^{1/2}}\right|_{E=E_n}. \tag{18}$$

The functions $\varphi_n(x)$ can be recognized as the so-called *Gamow states* [19, 21]. On the one hand, just like the *bona fide* energy eigenfunctions, they satisfy the differential equation

[5] More precisely, the bound is valid only outside the sectors $|\arg(E) + 2\pi/3| < \delta$ and $-\delta < \arg(E) < 0$, with $\delta > 0$ depending on $|E|$. As shown in section 2.3 and appendix D, these regions contain poles of $G^+(E; x, x')$ with arbitrarily large absolute values, where the inequality (16) is obviously false. One can, however, make δ arbitrarily small by taking $|E|$ sufficiently large.

[6] In writing (17) and (18) we have made use of the fact that the poles of $G^+(E; x, x')$ are simple, as can be explicitly checked by direct inspection.

$H\varphi_n(x) = E_n\varphi_n(x)$. On the other hand, the complex quantities E_n do not correspond to the eigenvalues of the self-adjoint Hamiltonian operator and, moreover, the Gamow states are neither normalizable (not even in the sense of generalized functions, because they diverge when $x \to \infty$) nor mutually orthogonal.

Using equation (17) and the fact that

$$\psi(x,t) = \int_{-\infty}^{+\infty} \mathrm{d}x' K^+(t;x,x')\psi(x',0) \qquad t \geqslant 0 \tag{19}$$

we can recast the non-decay amplitude $A(t;[\psi]) := \langle\psi|\mathrm{e}^{-\mathrm{i}Ht}|\psi\rangle$ in the form of a resonant-mode expansion:

$$A(t;[\psi]) = \sum_n \tilde{C}_n C_n \exp(-\mathrm{i}E_n t) \tag{20}$$

where

$$C_n := \int_{-\infty}^{+\infty} \mathrm{d}x\,\psi(x,0)\varphi_n(x) \qquad \tilde{C}_n := \int_{-\infty}^{+\infty} \mathrm{d}x\,\psi^*(x,0)\varphi_n(x). \tag{21}$$

Note that $\tilde{C}_n \neq C_n^*$ and $|\varphi_n(x)|^2 \sim \exp(F^{-1/2}\Gamma_n x^{1/2})$ as $x \to \infty$. It follows therefrom that the wavefunction $\psi(x,0)$ of the initial state must decrease sufficiently fast at infinity in order that the coefficients $C_n, \tilde{C}_n$ exist. This condition is fulfilled by $\psi(x,0) = \psi_B(x)$. This still leaves open the question of whether the series (20) converges. Here we shall *assume* that it does, at least in the l^2-topology.

2.3. *Poles of the Green function*

The unperturbed Green function $G_0^+(E;x,x')$ is an holomorphic function of E, so that the poles of $G^+(E;x,x')$ are all given by the zeros of $g(\lambda,E)$. Inserting the explicit form of $G_0^+(E;0,0)$ into equation (14) and noting that when $F = 0$ there is a bound state with energy $E_B = -\lambda^2/4$, we arrive at the following equation:

$$\mathrm{Ai}(-\varepsilon)\,\mathrm{Ci}^{(+)}(-\varepsilon) = \frac{1}{2\pi}(-\varepsilon_B)^{-1/2} \qquad \varepsilon_B := E_B F^{-2/3}. \tag{22}$$

For a given value of ε_B, equation (22) has an infinite number of solutions, all located in the lower half-plane. Some of them are shown in figure 1. They can be numbered according to their values in the limit $\varepsilon_B \to -\infty$, which corresponds to a very weak field ($F \to 0$) or a very strong attractive interaction ($\lambda \to \infty$). One of the poles approaches the negative real axis and behaves asymptotically as (see appendix B)

$$\varepsilon_0 \sim \varepsilon_B \left\{1 + \mathrm{i}\exp\left[-\tfrac{4}{3}(-\varepsilon_B)^{3/2}\right]\right\} \qquad \varepsilon_B \to -\infty. \tag{23}$$

Its real part corresponds to the energy of the (unique) bound state of the atom in the absence of a uniform field. Its imaginary part is half the decay rate of the atom via tunnelling through the potential barrier created by the external field.

The other poles approach the zeros[7] of $\mathrm{Ai}(-\varepsilon)$, which are real and located on the positive real axis,

$$\lim_{\varepsilon_B \to -\infty} \varepsilon_n = -a_n \qquad n \in \mathbb{N} \tag{24}$$

and of $\mathrm{Ci}^{(+)}(-\varepsilon)$,

$$\lim_{\varepsilon_B \to -\infty} \varepsilon_{-n} = -a_n\,\mathrm{e}^{-2\mathrm{i}\pi/3} \qquad n \in \mathbb{N}. \tag{25}$$

[7] Note that the rhs of equation (22) vanishes in the limit $\varepsilon_B \to -\infty$.

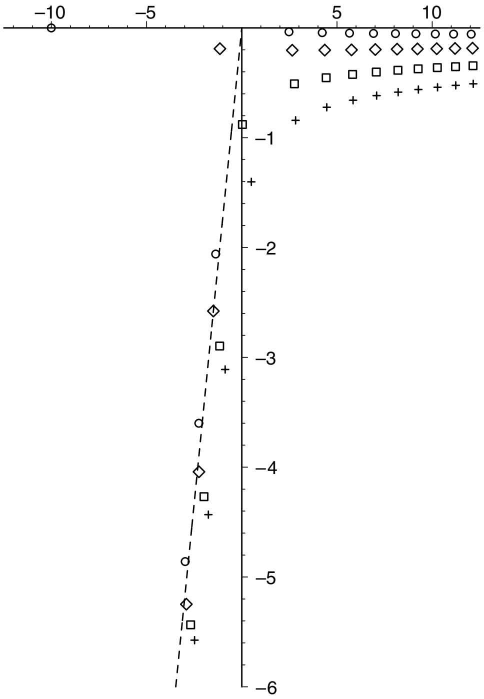

Figure 1. Poles of the Green function in the complex ε-plane (ε_{-3} to ε_9, clockwise) in the one-dimensional case: $\varepsilon_B = -10$ ($\circ$), $\varepsilon_B = -1$ ($\diamond$), $\varepsilon_B = -0.1$ ($\square$) and $\varepsilon_B = -0.01$ (+). The dashed line corresponds to the half-line $\arg(\varepsilon) = -2\pi/3$. (Angles appear distorted in this plot because the real and imaginary axes have different scales.)

In equation (24), a_n denotes the nth zero of $\mathrm{Ai}(z)$; equation (25) follows from the identity $\mathrm{Ci}^{(+)}(z) = 2\,\mathrm{e}^{\mathrm{i}\pi/6}\mathrm{Ai}(z\,\mathrm{e}^{2\mathrm{i}\pi/3})$ [18].

If the external field is very weak, but nonvanishing—i.e., $|\varepsilon_B| \gg 1$—then the poles ε_n with $n > 0$ exhibit a small negative imaginary part (see figure 1). However, while $\mathrm{Im}(\varepsilon_0)$ approaches zero exponentially fast as $\varepsilon_B \to -\infty$, one has $\mathrm{Im}(\varepsilon_n) \sim (-\varepsilon_B)^{-1}$ in the same limit (provided n is not very large, see appendix C). This means that the transient effects associated with the poles ε_n with $n > 0$—and *a fortiori* those associated with ε_n with $n < 0$—disappear much faster than the corresponding effects associated with the resonance ε_0.

Looking at figure 1, one can note that the imaginary parts of the first few poles ε_n with $n > 0$ have the same order of magnitude. This explains the short time oscillatory behaviour of $|A(t;[\psi])|^2$ observed in numerical studies [14–16] of the model (3) in the weak field regime: it is a consequence of the interference among the resonances associated with those poles. As a matter of fact, these resonances have a simple physical interpretation: when the external field F is turned on, it may excite the particle to a state of positive energy. Once excited, the particle is pushed to the positive x-direction by the field—recall that we are assuming $F > 0$—but it is scattered by the potential $V(x) = -\lambda\delta(x)$. Because the potential is strongly attractive as λ is very large, the transmission probability is small, so that the particle can bounce back and forth many times in the region to the left of the origin before it finally 'jumps over' the potential well.

Let us now examine the strong field regime $|\varepsilon_B| \ll 1$. In this case, as shown in figure 1, the decay rates $\{\Gamma_j | j \in \mathbb{Z}\}$ form a monotonic decreasing sequence, with $\lim_{j\to\infty} \Gamma_j = 0$, as shown in appendix D. Thus, in contrast with the class of potentials considered by Herbst [8], there does not exist a slower decaying resonance, which would eventually dominate the decay process. As a consequence, the decay is not asymptotically exponential: $\forall \alpha > 0, \lim_{t\to\infty} e^{\alpha t} |A(t; [\psi])|^2 = \infty$. By the way, strictly speaking this result actually holds true even in the weak field regime $|\varepsilon_B| \gg 1$, because $\lim_{n\to\infty} \Gamma_n = 0$ regardless of the value of ε_B (see appendix D). In the weak field case, however, one should have to wait an extremely long time until a deviation from the exponential decay $|A(t; [\psi])|^2 \sim \exp(-\Gamma_0 t)$ became appreciable. Besides, $|A(t; [\psi])|^2$ would be so small by then that such a deviation would be practically unobservable.

The crossover from the weak to the strong field regime occurs at $\varepsilon_B \sim -1$. At this value, $\mathrm{Im}(\varepsilon_0) \approx \mathrm{Im}(\varepsilon_1)$ (see figure 1)—an indication that the two mechanisms of decay discussed above become equally important.

3. The two- and three-dimensional cases

3.1. Retarded Green function

We can use the same strategy employed in section 2.1 to solve the D-dimensional version of equation (4), which reads

$$[E + \nabla^2 + \lambda\delta^{(D)}(\mathbf{x}) + Fx]G^+(E; \mathbf{x}, \mathbf{x}') = \delta^{(D)}(\mathbf{x} - \mathbf{x}') \tag{26}$$

where $\mathbf{x} = (x_1, \ldots, x_D) := (x, \mathbf{r})$ and $E \in \mathbb{C}$. Thus we can formally write $G^+(E; \mathbf{x}, \mathbf{x}')$ as in equation (13), in which $G_0^+(E; \mathbf{x}, \mathbf{x}')$ denotes the solution to equation (26) in the case $\lambda = 0$. The latter can be written as

$$G_0^+(E; \mathbf{x}, \mathbf{x}') = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \mathcal{G}_0^+(E, \mathbf{k}; x, x') \tag{27}$$

where $\mathcal{G}_0^+(E, \mathbf{k}; x, x')$ satisfies

$$\left(E - \mathbf{k}^2 + \frac{\partial^2}{\partial x^2} + Fx\right) \mathcal{G}_0^+(E, \mathbf{k}; x, x') = \delta(x - x'). \tag{28}$$

This has precisely the form of equation (4) with $\lambda = 0$ and $E \to E - \mathbf{k}^2$, the solution to which is given by equation (11). Inserting it into equation (27) we finally obtain

$$G_0^+(E; \mathbf{x}, \mathbf{x}') = -\pi F^{-1/3} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \exp(i\mathbf{k}\cdot(\mathbf{r} - \mathbf{r}'))\mathrm{Ai}(-\rho_-)\,\mathrm{Ci}^{(+)}(-\rho_+) \tag{29}$$

where now $\rho := F^{1/3}[x + (E - \mathbf{k}^2)/F]$ and $2\rho_\pm := \rho + \rho' \pm |\rho - \rho'|$.

3.2. Renormalization

In contrast with the one-dimensional case, the Green function is ill-defined at coincident points for $D \geqslant 2$. Indeed, after setting $\mathbf{x} = \mathbf{x}' = 0$ in equation (29) and performing the angular integration, we obtain

$$G_0^+(E; 0, 0) = -C_D F^{-1/3} \int_0^\infty \mathrm{Ai}(q)\,\mathrm{Ci}^{(+)}(q) k^{D-2}\,dk \tag{30}$$

where

$$C_D := \frac{2^{2-D}\pi^{(3-D)/2}}{\Gamma[(D-1)/2]} \qquad q := F^{-2/3}(\mathbf{k}^2 - E). \tag{31}$$

Since

$$\mathrm{Ai}(q)\,\mathrm{Ci}^{(+)}(q) \sim \frac{q^{-1/2}}{2\pi} \sim \frac{F^{1/3}}{2\pi k} \qquad \text{for} \quad k \to \infty \tag{32}$$

the integral in equation (30) turns out to be ultraviolet divergent in $D \geqslant 2$. In the two- and three-dimensional cases the divergence can be absorbed through a redefinition of the coupling parameter λ. To do this we follow the same procedure employed in [22]. Let us introduce a cutoff Λ in the upper limit of integration in equation (30) and add to the resulting expression the following integral:

$$I_D(\Lambda, \mu) := \frac{C_D}{2\pi} \int_0^\Lambda \frac{k^{D-2}\,\mathrm{d}k}{\sqrt{k^2 + \mu^2}} \tag{33}$$

which contains the arbitrary momentum scale $\mu > 0$. At the same time, we subtract $I_D(\Lambda, \mu)$ from λ^{-1} and define the renormalized coupling parameter λ_R as

$$[\lambda_R(\mu)]^{-1} := \lim_{\Lambda\to\infty} [\lambda^{-1} - I_D(\Lambda, \mu)] \tag{34}$$

where it is understood that λ depends on Λ in such a way that the limit exists. In this way, the denominator of the Krein formula (13) is replaced by an expression that is finite when the cutoff is removed:

$$\begin{aligned} \lim_{\Lambda\to\infty} g_D(\lambda, E) &= \frac{1}{\lambda_R} - \frac{C_D}{F^{1/3}} \int_0^\infty \left[\mathrm{Ai}(q)\,\mathrm{Ci}^{(+)}(q) - \frac{F^{1/3}}{2\pi\sqrt{k^2 + \mu^2}} \right] k^{D-2}\,\mathrm{d}k \\ &:= g_D(\lambda_R, \mu, E). \end{aligned} \tag{35}$$

In the next two subsections we shall analyse this expression separately in $D = 2$ and $D = 3$ dimensions. It turns out that the latter is simpler than the former, so we discuss it first.

3.3. *Three-dimensional case*

In $D = 3$ the integral in equation (35) can be computed in closed form[8] yielding ($\varepsilon := EF^{-2/3}$)

$$g_3(\lambda_R, \mu, E) = \frac{1}{\lambda_R} - \frac{\mu}{4\pi} - \frac{1}{4} F^{1/3} [\varepsilon \mathrm{Ai}(-\varepsilon)\,\mathrm{Ci}^{(+)}(-\varepsilon) + \mathrm{Ai}'(-\varepsilon)\,\mathrm{Ci}^{(+)\prime}(-\varepsilon)]. \tag{36}$$

Using the asymptotic expressions of the Airy functions for large argument [18], one can easily show that in the limit $F \to 0$ the expression above is reduced to

$$g_3(\lambda_R, \mu, E)|_{F=0} = \frac{1}{\lambda_R} - \frac{\mu}{4\pi} + \frac{\sqrt{-E}}{4\pi}. \tag{37}$$

Thus, provided $\lambda_R > 4\pi/\mu$, the quantity $g_3(\lambda_R, \mu, E)$ has a real zero given by

$$E_B = -\left[\mu - \frac{4\pi}{\lambda_R(\mu)}\right]^2 \tag{38}$$

which can be identified as the energy of the unique bound state of the system. It is worthwhile to remark that the bound state energy is a physical quantity and turns out to be independent of the arbitrary scale μ. From this physical requirement one can readily obtain the flow equation

[8] $\int y_1 y_2\,\mathrm{d}x = x y_1 y_2 - y_1' y_2'$ for any two solutions of the Airy equation $y'' - xy = 0$.

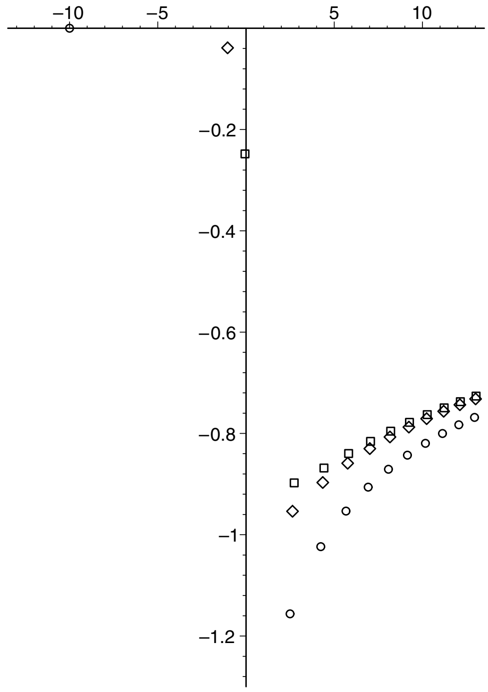

Figure 2. Poles of the Green function in the complex ε-plane (ε_0 to ε_{10}, from left to right) in the three-dimensional case: $\varepsilon_B = -10$ ($\circ$), $\varepsilon_B = -1$ ($\diamond$) and $\varepsilon_B = -0.1$ ($\square$).

for the renormalized *running* coupling parameter:

$$\lambda_R(\mu) = \frac{\lambda_R(\mu_0)}{1 + (\mu - \mu_0)[\lambda_R(\mu_0)/4\pi]} \tag{39}$$

which exhibits asymptotic freedom, i.e., $\lambda_R(\mu) \to 0$ as $\mu \to \infty$.

After setting $\varepsilon_B := F^{-2/3} E_B$ we can rewrite the resonance equation $g_3(\lambda_R, \mu, E) = 0$ in the form

$$\frac{1}{\pi}(-\varepsilon_B)^{1/2} + \varepsilon \mathrm{Ai}(-\varepsilon)\,\mathrm{Ci}^{(+)}(-\varepsilon) + \mathrm{Ai}'(-\varepsilon)\,\mathrm{Ci}^{(+)'}(-\varepsilon) = 0 \tag{40}$$

that generalizes equation (22) to the three-dimensional case. As in the one-dimensional case, equation (40) has a solution ε_0 that tends asymptotically to ε_B in the weak field regime (see appendix B):

$$\varepsilon_0 \sim \varepsilon_B \left\{ 1 + \frac{\mathrm{i}}{4}(-\varepsilon_B)^{-3/2} \exp\left[-\frac{4}{3}(-\varepsilon_B)^{3/2} \right] \right\} \qquad \varepsilon_B \to -\infty. \tag{41}$$

It has the same physical interpretation as its one-dimensional counterpart (see equation (23)).

In addition to ε_0, equation (40) has an infinite number of solutions. Some of them are shown in figure 2 for three different values of ε_B. Their distribution in the complex ε-plane bears some resemblance to the one-dimensional case (see figure 1); in particular, they approach asymptotically the half-lines $\arg(\varepsilon) = -2\pi/3$ and $\arg(\varepsilon) = 0$ (see appendix D). There are, however, two important differences:

(i) for fixed $n > 0$ we have that[9] $\lim_{\varepsilon_B \to -\infty} \mathrm{Im}(\varepsilon_n) \neq 0$, as shown in figure 2, which clearly exhibits that the larger $|\varepsilon_B|$ is the farther ε_n is from the real axis;

[9] Note that this fact is not in conflict with equation (D.6), which is valid under the condition that $|\varepsilon_n| \gg |\varepsilon_B|$.

(ii) there is no clear distinction between the weak and strong field regimes—Γ_0 is always smaller than Γ_1, even for $\varepsilon_B \to 0$.

The first difference has a simple geometric interpretation: in three dimensions a particle can avoid a localized obstacle by going around it. Hence, the field F can easily detach a particle with positive energy from a localized potential.

3.4. Two-dimensional case

Let us now finally discuss the two-dimensional case. Concerning this, it is important to realize that the integral of equation (35) is no longer expressible in closed form when $D = 2$. Here we shall content ourselves with deriving an asymptotic expression for ε_0 in the limit $F \to 0$. With this aim let us assume that $|\varepsilon_0|$ is large and close to the negative real half-axis. In this case—see equation (31)—$|q|$ is large and $|\arg(q)| \approx \pi$ for all $k \in [0, \infty)$, hence the following approximation is uniformly valid in the range of integration in equation (35) [18]:

$$\mathrm{Ai}(q)\,\mathrm{Ci}^{(+)}(q) \sim \frac{F^{1/3}}{2\pi\sqrt{k^2 - E}} \left\{ 1 + \frac{\mathrm{i}}{2} \exp\left[-\frac{4}{3} F^{-1} (k^2 - E)^{3/2} \right] \right\}. \tag{42}$$

Inserting this into equation (35) we obtain, in $D = 2$,

$$g_2(\lambda_R, \mu, E) \sim \frac{1}{\lambda_R} - \frac{1}{4\pi} \ln\left(-\frac{\mu^2}{E} \right) - \frac{\mathrm{i}}{4\pi} I_2(E) \tag{43}$$

where

$$I_2(E) := \int_0^\infty \frac{\mathrm{d}k}{\sqrt{k^2 - E}} \exp\left[-\frac{4}{3} F^{-1} (k^2 - E)^{3/2} \right]. \tag{44}$$

Consistently with our assumptions on E and F we can compute $I_2(E)$ using the saddle-point approximation and obtain

$$\begin{aligned} I_2(E) &\sim \int_0^\infty \frac{\mathrm{d}k}{\sqrt{-E}} \exp\left\{ -\frac{4}{3} F^{-1} \left[(-E)^{3/2} + \frac{3}{2} (-E)^{1/2} k^2 \right] \right\} \\ &= \sqrt{\frac{\pi F}{8}} (-E)^{-3/4} \exp\left[-\frac{4}{3} F^{-1} (-E)^{3/2} \right]. \end{aligned} \tag{45}$$

In the limit $F \to 0$, the integral $I_2(E)$ vanishes and $g_2(\lambda_R, \mu, E)$ has a single real and negative zero E_B, corresponding to the energy of the bound state in the absence of the external field:

$$E_B = -\mu^2 \exp\left[-\frac{4\pi}{\lambda_R(\mu)} \right]. \tag{46}$$

We note that in $D = 2$ a bound state exists—provided, of course, $F = 0$—even if the renormalized strength of the point-like potential is negative, in which case one could naively expect the potential to be repulsive.

As the bound state energy must be independent of the arbitrary scale μ, one can readily obtain the flow equation for the renormalized *running* coupling parameter, that now reads

$$\lambda_R(\mu) = \frac{\lambda_R(\mu_0)}{1 + [\lambda_R(\mu_0)/2\pi] \ln(\mu/\mu_0)} \tag{47}$$

leading again to asymptotic freedom.

Now, let us consider equation (43) in the case of a weak field F. Using equation (46), we can rewrite it as

$$g_2(E_B, E) \sim \frac{1}{4\pi} \left[\ln\left(\frac{E}{E_B} \right) - \mathrm{i} I_2(E) \right]. \tag{48}$$

An approximate solution to the equation $g_2(E_B, E) = 0$ is given by $E_0 = E_B\,[1 + \mathrm{i}I_2(E_B)]$; in terms of the dimensionless variable $\varepsilon = EF^{-2/3}$ we obtain (cf equation (45)),

$$\varepsilon_0 \sim \varepsilon_B \left\{ 1 + \mathrm{i}\sqrt{\frac{\pi}{8}}(-\varepsilon_B)^{-3/4} \exp\left[-\frac{4}{3}(-\varepsilon_B)^{3/2} \right] \right\} \qquad \varepsilon_B \to -\infty. \tag{49}$$

Moreover, as in the one- and three-dimensional cases, it is possible to show that an infinite number of resonances arise as solutions to the equation $g_2(\lambda_R, \mu, E) = 0$, approaching asymptotically the half-lines $\arg(E) = 0$ and $\arg(E) = -2\pi/3$ when $|E| \to \infty$.

4. Conclusions

In this paper, we have analysed the ionization of a very simple (though non-trivial) model atom subjected to the influence of a uniform static field. The model we have considered here is that of a one-electron atom in which the Coulomb interaction between the electron and the nucleus is replaced by an attractive short-range (in fact, point-like) interaction. We have analysed the problem in $D = 1, 2$ and 3 spatial dimensions. In spite of its simplicity and the fact that—due to the external field—the Hamiltonian is not bounded from below, the study of the present model is far from being academic as it allows us to grasp the basic features of the quantum dynamical behaviour of many realistic physical systems. In particular, its main prediction is a sensible deviation, in the strong field regime, from the naively expected exponential decay law of the survival probability of the bound state after the external field is turned on. Actually, more or less important deviations from the exponential decay law do occur even when the field is weak, specially in its short time behaviour, with the presence of oscillatory transient effects (which are more pronounced in $D = 1$). Such deviations are caused by the presence of a purely continuous spectrum and the appearance of an infinite number of resonances once the uniform field is switched on. Deviations from the exponential decay law are also expected for very large times; this, however, may be an artefact of the model studied here, since for more realistic potentials one can prove asymptotic exponential decay [8]. (On the other hand, as noted before, the survival probability would be so small when such deviations took place that they would be practically unobservable.)

An important development of the present investigation, which will be presented elsewhere, is the generalization of our analysis to the additional presence of a uniform magnetic field. In this way, it might eventually be possible to precisely evaluate the lifetimes of the so called *non-conducting* states—within the integer quantum Hall effect (IQHE) conventional terminology—and to explicitly verify the widely popular picture according to which the presence of impurities, described in the simplest way by point-like attractive wells, gives rise to the plateaux formation in the IQHE [23].

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Appendix A

In this appendix we shall sketch the proof of the bound (16) on $G^+(E; x, x)$ in $D = 1$. Let us first examine the asymptotic behaviour of $G^+(E; x, x')$ for $|E| \to \infty$ in the sector

$-\pi < \arg(E) < -2\pi/3$. Since $\arg(\rho) \to \arg(E)$ as $|E| \to \infty$ (cf equation (7)) we have $|\arg(-\rho)| = |\arg(e^{i\pi}\rho)| < \pi/3$ for $|E|$ large enough, so that [18]

$$\mathrm{Ai}(-\rho) \sim \tfrac{1}{2}\pi^{-1/2}(-\rho)^{-1/4}\exp\left[-\tfrac{2}{3}(-\rho)^{3/2}\right] \tag{A.1}$$

$$\mathrm{Ci}^{(+)}(-\rho) \sim \pi^{-1/2}(-\rho)^{-1/4}\left\{\exp\left[\tfrac{2}{3}(-\rho)^{3/2}\right] + \tfrac{1}{2}\,\mathrm{i}\exp\left[-\tfrac{2}{3}(-\rho)^{3/2}\right]\right\}. \tag{A.2}$$

Inserting equations (A.1) and (A.2) into equation (11), and dropping the second term in curly brackets in equation (A.2) since it is negligible compared to the first one, we obtain

$$G_0^+(E;x,x') \sim \tfrac{1}{2}\,\mathrm{i}F^{-1/3}(\rho_-\rho_+)^{-1/4}\exp\left[\tfrac{2}{3}\mathrm{i}\left(\rho_-^{3/2} - \rho_+^{3/2}\right)\right]. \tag{A.3}$$

In addition, taking equation (7) into account, we have

$$\rho^{3/2} \sim F^{-1}E^{3/2} + \tfrac{3}{2}E^{1/2}x \qquad |E| \to \infty \tag{A.4}$$

so that

$$G_0^+(E;x,x') \sim \tfrac{1}{2}\mathrm{i}E^{-1/2}\exp\left(-\mathrm{i}E^{1/2}|x-x'|\right). \tag{A.5}$$

Inserting this expression into equation (13) and using the fact that $\mathrm{Im}(E^{1/2}) < 0$, one can easily show that there is a positive constant C such that $|G^+(E;x,x')| < C|E|^{-1/2}$ for $|E|$ sufficiently large and $-\pi < \arg(E) < -2\pi/3$. Note that equation (16) is a trivial consequence of this inequality.

Let us now examine the asymptotic behaviour of $G^+(E;x,x')$ in the sector $-2\pi/3 < \arg(E) < 0$. For this purpose it is convenient to rewrite equation (13) as

$$G^+(E;x,x') = \frac{G_0^+(E;x,x') + \lambda R(E;x,x')}{1 + \lambda G_0^+(E;0,0)} \tag{A.6}$$

where

$$R(E;x,x') := G_0^+(E;0,0)G_0^+(E;x,x') - G_0^+(E;x,0)G_0^+(E;0,x'). \tag{A.7}$$

Again, since $\arg(\rho) \to \arg(E)$ as $|E| \to \infty$, we have $|\arg(\rho)| < 2\pi/3$ for $|E|$ large enough, in which case we have [18]

$$\mathrm{Ai}(-\rho) \sim \pi^{-1/2}\rho^{-1/4}\sin\left(\tfrac{2}{3}\rho^{3/2} + \tfrac{1}{4}\pi\right) \tag{A.8}$$

$$\mathrm{Ci}^{(+)}(-\rho) \sim \pi^{-1/2}\rho^{-1/4}\exp\left[\mathrm{i}\left(\tfrac{2}{3}\rho^{3/2} + \tfrac{1}{4}\pi\right)\right] \tag{A.9}$$

so that

$$G_0^+(E;x,x') \sim \tfrac{1}{2}\mathrm{i}F^{-1/3}(\rho_-\rho_+)^{-1/4}\left\{\mathrm{i}\exp\left[\tfrac{2}{3}\mathrm{i}\left(\rho_+^{3/2} + \rho_-^{3/2}\right)\right] - \exp\left[\tfrac{2}{3}\mathrm{i}\left(\rho_+^{3/2} - \rho_-^{3/2}\right)\right]\right\}. \tag{A.10}$$

Using (A.4) and neglecting the second term in curly brackets we obtain

$$G_0^+(E;x,x') \sim -\tfrac{1}{2}E^{-1/2}\exp\left\{\mathrm{i}\left[\tfrac{4}{3}E^{3/2}F^{-1} + E^{1/2}(x+x')\right]\right\}. \tag{A.11}$$

The asymptotic behaviour of $R(E; x, x')$ depends upon the signs of x and x'. Let us first consider the case $x, x' \leqslant 0$. If we insert equation (11) into equation (A.7), and use the identity $f(\rho)f(\rho') = f(\rho_-)f(\rho_+)$, we obtain ($\varepsilon := EF^{-2/3}$)

$$R(E; x, x') = \pi^2 F^{-2/3}\mathrm{Ai}(-\rho_-)\,\mathrm{Ci}^{(+)}(-\varepsilon)[\mathrm{Ai}(-\varepsilon)\,\mathrm{Ci}^{(+)}(-\rho_+) - \mathrm{Ai}(-\rho_+)\,\mathrm{Ci}^{(+)}(-\varepsilon)]. \tag{A.12}$$

Furthemore, inserting equations (A.8) and (A.9) into equation (A.12) and using equation (A.4) we obtain

$$R(E; x, x') \sim -\frac{1}{2E}\exp\left[\mathrm{i}\left(\frac{4}{3}E^{3/2}F^{-1} + E^{1/2}x_-\right)\right]\sin(E^{1/2}x_+) \qquad x, x' \leqslant 0 \tag{A.13}$$

where subdominant terms have been dropped. A similar analysis shows that

$$R(E; x, x') \sim \frac{1}{2E}\exp\left[\mathrm{i}\left(\frac{4}{3}E^{3/2}F^{-1} + E^{1/2}x_+\right)\right]\sin(E^{1/2}x_-) \qquad x, x' \geqslant 0 \tag{A.14}$$

and that $R(E; x, x') \equiv 0$ for $x \leqslant 0 \leqslant x'$ or $x' \leqslant 0 \leqslant x$. Replacing $G_0^+(E; x, x')$ and $R(E; x, x')$ in equation (A.6) with their asymptotic expressions, and using the inequalities $|\sin z| \leqslant \exp(|\mathrm{Im}\, z|)$, $|\mathrm{Im}\, E^{1/2}| \leqslant |E|^{1/2}$ and $|x \pm x'| \leqslant |x| + |x'|$, one can easily derive the bound of equation (16).

Appendix B

In this appendix we derive the asymptotic expression of ε_0 in the weak field limit. Let us first consider the one-dimensional case. If we assume that $|\varepsilon| \gg 1$ and $\arg(\varepsilon) \approx -\pi$, then we may use the asymptotic expressions (A.1) and (A.2) for the Airy functions Ai and $\mathrm{Ci}^{(+)}$. Equation (22) then becomes

$$(-\varepsilon)^{-1/2}\left\{1 + \tfrac{1}{2}\mathrm{i}\exp\left[-\tfrac{4}{3}(-\varepsilon)^{3/2}\right]\right\} \approx (-\varepsilon_B)^{-1/2}. \tag{B.1}$$

This equation can be solved iteratively. As a first approximation, one may neglect the second term in square brackets, thus obtaining $\varepsilon_0 \approx \varepsilon_B$. In order to obtain the imaginary part of ε_0 one must iterate once more: replacing ε in the exponential with ε_B and solving the resulting equation, one finds

$$\varepsilon_0 \sim \varepsilon_B\left\{1 + \mathrm{i}\exp\left[-\tfrac{4}{3}(-\varepsilon_B)^{3/2}\right]\right\} \qquad \varepsilon_B \to -\infty \quad D = 1. \tag{B.2}$$

One can derive a systematic expansion in powers of ε_B if one includes more and more terms in the asymptotic expansion of the Airy functions. In particular, the real part of the resulting expansion for $E_0 = F^{2/3}\varepsilon_0$ agrees with the Rayleigh–Schrödinger perturbation series for the bound state energy when the external field is treated as a perturbation [14, 16].

Following the same strategy, we can approximate equation (40) of the three-dimensional case by

$$(-\varepsilon_B)^{1/2} - (-\varepsilon)^{1/2} - \frac{\mathrm{i}}{8\varepsilon}\exp\left[-\frac{4}{3}(-\varepsilon)^{3/2}\right] = 0. \tag{B.3}$$

We can obtain an approximate solution to this equation using the iterative method employed above. This way, we finally arrive at the result displayed in equation (41). The result for the two-dimensional case is worked out in subsection 3.4 and is given in equation (49).

Appendix C

In this appendix we derive the asymptotic behaviour of ε_1 in the weak field limit, $\varepsilon_B \to -\infty$. As discussed in subsection 2.3, in that limit the rhs of equation (22) vanishes so that, to the lowest order, one has $\varepsilon_1 \approx -a_1$, where $a_1 = -2.338\,10\ldots$ is the smallest (in absolute value) zero of $\mathrm{Ai}(z)$. In order to obtain a more refined approximation, valid for a finite though large value of $|\varepsilon_B|$, we expand the lhs of equation (22) in powers of $x = \varepsilon + a_1$ and, assuming that $|x| \ll 1$, we truncate the series and solve the resulting polynomial equation in x. The first non-trivial correction to the imaginary part of ε_1 is obtained when one truncates the series at $O(x^3)$. In doing so, equation (22) is then approximated by a quadratic equation in x,

$$ax^2 + bx + c = 0 \tag{C.1}$$

where $a = \mathrm{Ai}'(a_1)\,\mathrm{Ci}^{(+)'}(a_1)$, $b = -\mathrm{Ai}'(a_1)\mathrm{Bi}(a_1)$, and $c = -(1/2\pi)(-\varepsilon_B)^{-1/2} \ll 1$. Of the two solutions to equation (C.1), $x_\pm = \left(-b \pm \sqrt{b^2 - 4ac}\right)\big/2a$, the one with the minus sign must be discarded, as it violates the condition that $x \to 0$ as $\varepsilon_B \to -\infty$ $(c \to 0)$. Expanding x_+ in powers of c, we obtain

$$x_+ = -\frac{c}{b} - \frac{ac^2}{b^3} + O(c^3). \tag{C.2}$$

Substituting a, b and c with their explicit expressions, we finally obtain

$$\varepsilon_1 = -a_1 + \frac{1}{\mathrm{Ai}'(a_1)\mathrm{Bi}(a_1)}\frac{(-\varepsilon_B)^{-1/2}}{2\pi} + \frac{\mathrm{Ci}^{(+)'}(a_1)}{\mathrm{Ai}'(a_1)^2\mathrm{Bi}(a_1)^3}\frac{(-\varepsilon_B)^{-1}}{4\pi^2} + O\left[(-\varepsilon_B)^{-3/2}\right] \qquad \varepsilon_B \to -\infty. \tag{C.3}$$

An important consequence of this result is that $\mathrm{Im}(\varepsilon_1) \sim (-\varepsilon_B)^{-1}$ for $\varepsilon_B \to -\infty$.

One could be tempted to apply the reasoning above to any ε_n, $n \in \mathbb{N}$. However, there is an important caveat: the rhs of equation (C.2) is a good approximation to x_+ only if $|ac/b^2| \ll 1$, or

$$\left|\frac{a}{b^2}\right| = \left|\frac{\mathrm{Ci}^{(+)'}(a_n)}{\mathrm{Ai}'(a_n)\mathrm{Bi}(a_n)^2}\right| \ll |c| = 2\pi(-\varepsilon_B)^{1/2}. \tag{C.4}$$

Using the asymptotic expressions of the Airy functions and of a_n—the nth zero of $\mathrm{Ai}(z)$ [18]—one can show that $|a/b^2| \sim \pi(-a_n)^{1/2}$. Hence, equation (C.3) is also valid for ε_n, $n > 1$ (with the obvious substitution $a_1 \to a_n$), provided $|a_n| \ll |\varepsilon_B|$. (See appendix D for the asymptotic behaviour of ε_n when $|a_n| \gg |\varepsilon_B|$.)

Appendix D

In this appendix we derive the asymptotic behaviour of the resonances ε_n, $n \neq 0$, which are located very far from the origin in the complex ε-plane. Let us first discuss the one-dimensional case. Assuming that $|\varepsilon| \gg 1$ and $\theta := \arg(\varepsilon) \approx 0$, we are allowed to use the asymptotic expressions (A.8) and (A.9) with the aim of approximating equation (22) by

$$\varepsilon^{-1/2}\left[\exp\left(\tfrac{4}{3}\mathrm{i}\varepsilon^{3/2}\right) + \mathrm{i}\right] \approx (-\varepsilon_B)^{-1/2}. \tag{D.1}$$

If we further assume that $|\varepsilon| \gg |\varepsilon_B|$, we may neglect the second term in square brackets; the resulting complex equation is then equivalent to the following pair of real equations:

$$\frac{4}{3}|\varepsilon|^{3/2}\sin\frac{3\theta}{2} \approx -\frac{1}{2}\ln\left|\frac{\varepsilon}{\varepsilon_B}\right| \qquad \frac{4}{3}|\varepsilon|^{3/2}\cos\frac{3\theta}{2} \approx \frac{\theta}{2} + 2n\pi \qquad n \in \mathbb{N}. \tag{D.2}$$

Assuming $n \gg 1$ and $|\varepsilon|$ large, one can easily solve these equations, obtaining

$$\theta \sim -\frac{1}{4}|\varepsilon|^{-3/2} \ln\left|\frac{\varepsilon}{\varepsilon_B}\right| \qquad |\varepsilon| \sim s_n := \left(\frac{3n\pi}{2}\right)^{2/3}. \tag{D.3}$$

Since $|\theta| \ll 1$, we can write $\varepsilon = |\varepsilon|\,\mathrm{e}^{\mathrm{i}\theta} \approx |\varepsilon|(1 + \mathrm{i}\theta)$, so that

$$\varepsilon_n \sim s_n - \frac{\mathrm{i}}{4} s_n^{-1/2} \ln\left|\frac{s_n}{\varepsilon_B}\right| \qquad n \gg 1 \quad D = 1. \tag{D.4}$$

Next we consider the case $|\varepsilon| \gg 1$ and $\theta := \arg(\varepsilon) \approx -2\pi/3$. Using the very same approximations we readily come to the following estimate

$$\varepsilon_{-n} \sim \mathrm{e}^{-2\mathrm{i}\pi/3}\left\{s_n + \frac{\mathrm{i}}{4} s_n^{-1/2} \ln\left|\frac{s_n}{\varepsilon_B}\right|\right\} \qquad n \gg 1 \quad D = 1. \tag{D.5}$$

A straightforward generalization of the above treatments to the basic resonance equation (40) in the three-dimensional case eventually leads to the following asymptotic expressions:

$$\varepsilon_n \sim s_n - \tfrac{1}{2}\mathrm{i} s_n^{-1/2} \ln\left(4 s_n^{3/2}\right) \qquad \varepsilon_{-n} \sim \mathrm{e}^{-2\mathrm{i}\pi/3}\varepsilon_n^* \qquad n \gg 1 \quad D = 3. \tag{D.6}$$

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